LECTURES ON DISORDERED MODELS - EXERCISE ON UNIQUENESS OF THE GROUND STATE IN THE TWO-DIMENSIONAL RANDOM-FIELD ISING MODEL

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1. General notation

1.1. **Lattices.** We consider the lattice $\mathbb{Z}^d$ in dimension $d \geq 1$. Given two vertices $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if they are nearest-neighbours in $\mathbb{Z}^d$.

Given an integer $L \geq 0$, we consider the box $\Lambda_L := \{-L, \ldots, L\}^d \subseteq \mathbb{Z}^d$. Denote by $\partial \Lambda_L := \Lambda_{L+1} \setminus \Lambda_L$ its external vertex boundary and by $|\Lambda_L|$ its cardinality (i.e., $|\Lambda_L| = (2L+1)^d$).

Given a measurable set $A \subseteq \mathbb{R}$, we denote its Lebesgue measure by $\mathrm{Leb}(A)$.

1.2. **Ground state of the disordered Ising model.** We introduce the following notation for the *configurations* of the Ising model in the box Λ_L with + and - boundary conditions, respectively,

$$\mathcal{S}_L^+ := \left\{ \sigma : \mathbb{Z}^d \to \{-1, 1\} \text{ with } \sigma_v = 1 \text{ for } v \notin \Lambda_L \right\},$$
$$\mathcal{S}_L^- := \left\{ \sigma : \mathbb{Z}^d \to \{-1, 1\} \text{ with } \sigma_v = -1 \text{ for } v \notin \Lambda_L \right\}.$$

An *external field* is a function $h : \mathbb{Z}^d \to \mathbb{R}$. We will later take this function to be random, in which case we will denote it by ζ. Given a vertex $y \in \mathbb{Z}^d$ and an external field $h : \mathbb{Z}^d \to \mathbb{R}$, we denote by $\tau_y h : \mathbb{Z}^d \to \mathbb{R}$ the shifted field defined by $(\tau_y h)_x := h(x+y)$.

For each external field $h : \mathbb{Z}^d \to \mathbb{R}$, we define the *energy of the finite-volume ground states* of the Ising model with + and - boundary conditions and external field h by

$$F_L^+(h) := \sup_{\sigma \in \mathcal{S}_L^+} \left(\sum_{\substack{x \sim y \\ \{x,y\} \cap \Lambda_L \neq \emptyset}} \sigma_x \sigma_y + \sum_{x \in \Lambda_L} h_x \sigma_x \right) \quad \text{and} \quad F_L^-(h) := \sup_{\sigma \in \mathcal{S}_L^-} \left(\sum_{\substack{x \sim y \\ \{x,y\} \cap \Lambda_L \neq \emptyset}} \sigma_x \sigma_y + \sum_{x \in \Lambda_L} h_x \sigma_x \right)$$

and denote the energy difference by

$$F_L(h) := F_L^+(h) - F_L^-(h).$$

Note that, for almost every value of the field h on Λ_L, there are *unique* maximisers in the definitions of $F_L^+(h)$ and $F_L^-(h)$. We denote them by $\sigma_L^+(h)$ and $\sigma_L^-(h)$, respectively (the finite-volume ground states).

2. The Imry-Ma phenomenon

2.1. **Preliminaries: An analysis lemma.** For each pair of Lipschitz, convex functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$, we introduce the set (of points with δ-diverging derivatives)

$$\mathrm{Div}(F_1, F_2, \delta) := \{ t \in \mathbb{R} : F_1 \text{ and } F_2 \text{ are differentiable at } t \text{ and } |F_1'(t) - F_2'(t)| > \delta \}.$$

Exercise 1. *Show that there exists a constant $C > 0$ such that for each pair of convex and 1-Lipschitz functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ satisfying $|F_1 - F_2| \leq 1$ and each $\delta > 0$, one has the upper bound,*

$$\mathrm{Leb}\left(\mathrm{Div}(F_1, F_2, \delta)\right) \leq \frac{C}{\delta^2}. \tag{2.1}$$

2.2. **The Imry–Ma phenomenon.**

Exercise 2. *In this guided exercise we explain how Exercise 1 may be used to deduce the uniqueness of the ground state in the two-dimensional random-field Ising model. The exercise is loosely based on [1], where a quantitative bound is achieved using an additional fractal (Mandelbrot) percolation.*

Fix $\lambda \in (0, \infty)$. Let $(\zeta_x)_{x \in \mathbb{Z}^2}$ be independent Gaussian random variables with expectation 0 and variance λ^2.

(1) (Convexity, differentiability and deterministic bound) Show the following properties of F_L^+, F_L^-, F_L:
  (i) The functions $h \mapsto F_L^+(h)$ and $h \mapsto F_L^-(h)$ are convex.

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(ii) The functions $h \mapsto F_L^+(h)$ and $h \mapsto F_L^-(h)$ are differentiable almost everywhere and for every $x \in \Lambda_L$ and almost every value of h on Λ_L,

$$\frac{\partial F_L^+}{\partial h_x}(h) = \sigma_{L,x}^+(h) \quad \text{and} \quad \frac{\partial F_L^-}{\partial h_x}(h) = \sigma_{L,x}^-(h).$$

(iii) For any external field $h : \Lambda_L \to \mathbb{R}$,

$$|F_L(h)| \le 2\,|\partial \Lambda_L|\,.$$

(2) (Extremal boundary conditions) Show that, for almost every ζ and every $x \in \mathbb{Z}^d$,

$$\begin{cases} \sigma_{L,x}^-(\zeta) \le \sigma_{L,x}^+(\zeta), \\ \sigma_{L+1,x}^-(\zeta) \ge \sigma_{L,x}^-(\zeta) \\ \sigma_{L+1,x}^+(\zeta) \le \sigma_{L,x}^+(\zeta). \end{cases}$$

(3) (Convergence and translation covariance) For almost every ζ, deduce that for every $x \in \Lambda_L$,

$$\begin{cases} \sigma_{L,x}^-(\zeta) \underset{L\to\infty}{\longrightarrow} \sigma_x^-(\zeta), \\ \sigma_{L,x}^+(\zeta) \underset{L\to\infty}{\longrightarrow} \sigma_x^+(\zeta), \end{cases}$$

(where σ^-, σ^+ are defined as the limiting configurations) and, for every $y \in \mathbb{Z}^d$,

$$\sigma_y^-(\zeta) = \sigma_0^-(\tau_y \zeta) \quad \text{and} \quad \sigma_y^+(\zeta) = \sigma_0^+(\tau_y \zeta).$$

(4) (Magnetisation from energy) Let $1_{\Lambda_L} : \mathbb{Z}^d \to \{0,1\}$ be the indicator function of Λ_L. We set

$$\frac{\partial F_L}{\partial \hat{h}_L}(h) := \lim_{\delta\to 0} \frac{F_L(h + \delta 1_{\Lambda_L}) - F_L(h)}{\delta}.$$

Show that the following identity holds almost surely,

$$\frac{\partial F_L}{\partial \hat{h}_L}(\zeta) = \sum_{x\in\Lambda_L} \left(\sigma_{L,x}^+(\zeta) - \sigma_{L,x}^-(\zeta)\right).$$

(5) (Main bound: high density of uniqueness points) Assume that the dimension is $d = 2$. Deduce, using Exercise 1, that, for any $\delta > 0$,

$$\liminf_{L\to\infty} \mathbb{P}\left[\frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \left(\sigma_{L,x}^+(\zeta) - \sigma_{L,x}^-(\zeta)\right) < \delta\right] > 0.$$

(6) (Uniqueness of the ground state) Still assume that the dimension is $d = 2$. Deduce from the previous questions and the ergodic theorem that, for almost every ζ,

$$\sigma^-(\zeta) = \sigma^+(\zeta).$$

Hints:

- Questions 2 and 3: for $y \in \mathbb{Z}^d$, denote by $\sigma_{y+\Lambda_L}^+$ and $\sigma_{y+\Lambda_L}^-$ the finite-volume ground states of the Ising model in the box $(y + \Lambda_L)$ with + and - boundary conditions, respectively. Show that if $(y + \Lambda_L) \subseteq (y' + \Lambda_{L'})$ then $\sigma_{y+\Lambda_L}^+ \ge \sigma_{y'+\Lambda_{L'}}^+$ and $\sigma_{y+\Lambda_L}^- \le \sigma_{y'+\Lambda_{L'}}^-$.
- Question 5: We may use the following property of the Gaussian variables: if we denote by

$$\hat{\zeta}_L := \frac{1}{|\Lambda_L|} \sum_{x\in\Lambda_L} \zeta_x \quad \text{and} \quad \zeta_L^\perp := \zeta - \hat{\zeta}_L,$$

then the random variable $\hat{\zeta}_L$ and the random vector $\zeta_L^\perp$ are independent. Then fix a realization of $\zeta_L^\perp$ and apply a suitably rescaled version of Exercise 1 with the functions

$$\hat{\zeta}_L \to F_L^+(\hat{\zeta}_L, \zeta_L^\perp) \quad \text{and} \quad \hat{\zeta}_L \to F_L^-(\hat{\zeta}_L, \zeta_L^\perp).$$

One also needs the fact that the standard Gaussian distribution has full support on $\mathbb{R}$.

References

[1] Paul Dario, Matan Harel, and Ron Peled. "Quantitative disorder effects in low-dimensional spin systems." Communications in Mathematical Physics 405, no. 9 (2024): 212.