Opinion dynamics on complex networks

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March 22nd, 2025

Modeling opinion in social networks

- We model individuals as vertices on a marked directed graph $G = (V, E; \mathscr{A})$.
- An edge from vertex i to vertex j, (i, j), is interpreted as:

  "individual j **listens** to individual i".
- Individuals hold **opinions** about a given topic.
- Opinions take values on the interval $[-1, 1]$.
- There may be an **external media** that broadcasts a variety of opinions.
- At each time step $t = 1, 2, \ldots$, each individual **listens** to the opinions of all its inbound neighbors and those in the media, and then updates her own opinion.
- Individuals weigh the opinions they listen to in a personalized way, and may also control what media they listen to.

Model parameters: vertex attributes

- Let $(c(i,1), c(i,2), \ldots c(i,n)) \geq 0$ be the vector of weights for her neighbors' opinions; $c(i,k) \equiv 0$ if $(k,i) \notin E$ and $c(i,i) \equiv 0$.
- Weights are assumed to satisfy:

$$\sum_{j=1}^{n} c(i,j) = c \leq 1 \qquad \text{if } d_i^- = \sum_{j=1}^{n} 1(j \to i) > 0.$$

- Individuals have an **internal opinion** $q_i \in [-1, 1]$.
- The internal opinion remains static throughout the process, and may influence its dynamics.
- We call a vertex i with $d_i^- = 0$ a **stubborn agent**.

Model parameters: vertex attributes

- Each vertex $i \in V$ in the graph has a **mark** $\mathbf{x}_i$.
- Vertex marks usually include their in-degree and out-degree, but they can also include many other vertex attributes.
- In our model, marks include:
  - Internal opinion
  - Community label
  - Amount of trust given to each inbound neighbor
- Vertex marks are assumed to take values on a Polish space $\mathcal{S}$.
- We equip $\mathcal{S}$ with a metric ρ.

Model parameters: external media

- Let $W_i^{(t)}$ denote the external media signal received by individual i at time t, $t = 0, 1, 2, \ldots$.
- The media signals $\{W_i^{(t)} : t \geq 0\}$ are i.i.d. given $\mathbf{x}_i$ and the $\{W_i^{(t)} : i \in V, t \geq 0\}$ are conditionally independent given $\{\mathbf{x}_i : i \in V\}$.
- Media signals satisfy

$$|W_i^{(t)}| \leq d + c - \sum_{j=1}^{n} c(i, j),$$

for some $d \in (0, 1)$.

- Let $\nu(\mathbf{x}_i)$ denote the distribution of $W_i^{(0)}$.
- Let $R_i^{(t)}$ denote the **opinion** of individual i at time t.
- **Extension:** multiple topics make $\{R_i^{(t)}, W_i^{(t)}\}$ vectors.

The Friedkin-Johnsen model

- The **Friedkin-Johnsen** ('90) model is widely used in the social sciences for modeling opinions.
- All individuals in the graph $G = (V, E; \mathscr{A})$ update their opinions simultaneously at step $t+1$ according to the recursion:

$$R_i^{(t+1)} = \sum_{j=1}^{n} c(i,j) R_j^{(t)} + W_i^{(t)} + (1 - c - d) R_i^{(t)}, \qquad i \in V.$$

- **Special cases:**
  - $d_i^- \geq 1$ for all $i \in V \longrightarrow$ no stubborn agents
  - $c + d = 1 \longrightarrow$ no memory
  - $\{W_i^{(t)} : t \geq 0\}$ independent of $\mathbf{x}_i \longrightarrow$ pure noise
  - $\{W_i^{(t)} : t \geq 0\} \sim \nu(\mathbf{x}_i) \longrightarrow$ media signal that depends on individual's attributes

Matrix representation for the Friedkin-Johnsen model

- Explicit computation gives that if we let $\mathbf{W}^{(t)} = (W_1^{(t)}, \ldots, W_n^{(t)})'$, then

$$\mathbf{R}^{(t)} = \sum_{k=0}^{t-1} \sum_{s=0}^{k} a_{s,k} C^s \mathbf{W}^{(t-k)} + \sum_{s=0}^{t} a_{s,t} C^s \mathbf{R}^{(0)}$$

  for some matrix $C \in \mathbb{R}^{n \times n}$ and coefficients $\{a_{s,k}\}$.
- The matrix $C = (c(i,j))$ contains the weights each vertex assigns to its neighbors.

Goals for the model

- We want a model for the evolution of opinions on a social network that can predict complex behavior.
- The type of graphs covered in the analysis should be able to model real-world social networks.
- We want to model phenomena known as *confirmation bias* and *selective exposure*.
- The model should exhibit polarization under strong biases.
- **Goal:** explain when consensus is possible and quantify the potential of various depolarizing interventions.

The DeGroot model

- The **DeGroot** ('75) model does not include external media.
- Individuals update their opinion, synchronously or asynchronously, based only on their neighbors' opinions according to the recursion:

$$R_i^{(t+1)} = \sum_{j=1}^{n} c(i,j) R_j^{(t)} + (1-c) R_i^{(t)}, \qquad i \in V$$

- Provided the matrix of weights $C = (c(i,j))$ is irreducible and aperiodic, this model is known to achieve **consensus**, since

$$\mathbf{R}^{(t+1)} = C\mathbf{R}^{(t)} = C^{t+1}\mathbf{R}^{(0)},$$

so by the Perron-Frobenius theorem,

$$C^t \to \Pi, \qquad t \to \infty$$

where Π is a stochastic matrix with all its rows equal to each other.

Markov chain on a fixed graph

- The opinion model

$$R_i^{(t+1)} = \sum_{j=1}^{n} c(i,j) R_j^{(t)} + W_i^{(t)} + (1-c-d) R_i^{(t)}, \qquad i \in V,$$

  on a marked directed graph $G = (V, E; \mathscr{A})$ defines a **Markov chain** on $\mathbb{R}^{|V|}$.
- Let $\mathbf{R}^{(t)} = (R_1^{(t)}, \dots, R_{|V|}^{(t)})$.
- **Theorem:** (Fraiman-Lin-OC '22) Suppose G is locally finite and $d > 0$. Then, there exists a random vector $\mathbf{R}$ such that

$$\mathbf{R}^{(t)} \Rightarrow \mathbf{R}, \qquad t \to \infty.$$

Typical behavior

- Let $\mathbf{R} = (R_1, \ldots, R_{|V|})$ be the vector of stationary opinions.
- **Goal:** describe the distribution of R_I, where I is uniformly chosen in V.
- R_I represents the *typical* opinion of an individual in the network.
- The distribution of R_I also describes the proportion of individuals in the graph G having opinions in $A \subseteq [-1, 1]$, i.e.,

$$P(R_I \in A|G) = \frac{1}{|V|} \sum_{i \in V} 1(R_i \in A).$$

- In small graphs the distribution of $\mathbf{R}$ will greatly depend on G.
- On large graphs, only the statistical properties of the graph matter.

Modeling large graphs using random graph theory

- So far, we have thought of the graph G representing the social network as fixed.
- **Idea:** think of G as a realization from some random graph model.
- **Question:** can we find a random graph model that could have produced the specific graph G?

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- **Answer:** depends on how many properties of G we need to model....
- "First order" properties:
  - Degree distribution(s) (scale free property)
  - Connectivity
  - Typical distances (small world phenomenon)
  - Community structure

Random graph models

- "First order" properties are easy to model.
- Models that describe a "snapshot" of a graph are called **static**.
- Models that describe the evolution of a graph as it grows are called **dynamic**.
- Static models that can model first order properties include:
  - Erdős-Rényi model
  - Chung-Lu or expected given degree model
  - Norros-Reittu or Poissonian random graph
  - Generalized random graph
  - Configuration model
  - Stochastic block model
- Dynamic models include the Albert-Barabási or preferential attachment model and its generalizations.
- Our focus from now on will be on static models.

Opinion dynamics on random graphs

- From now on, assume $\{G_n : n \geq 1\}$ is a sequence of marked directed random graphs $G_n = (V_n, E_n; \mathscr{A}_n)$.
- Assume the graphs are **sparse** (i.e., the expected degrees are bounded).
- **Note:** Results are also available for semi-sparse and dense graphs.
- Suppose $\{G_n : n \geq 1\}$ converges in the **local weak sense** to a marked Galton-Watson tree (single or multi-type).
- We will consider the case $d > 0$ first.

Opinion model on random graphs

- We start by realizing the graph $G_n = (V_n, E_n; \mathscr{A}_n)$.
- Assume it has K communities (e.g., G_n is a dSBM).
- To construct the weights for the opinion of the neighbors, for each edge $(j, i) \in E_n$ we sample

$$B_{ij} \sim G_{J_i, J_j}$$

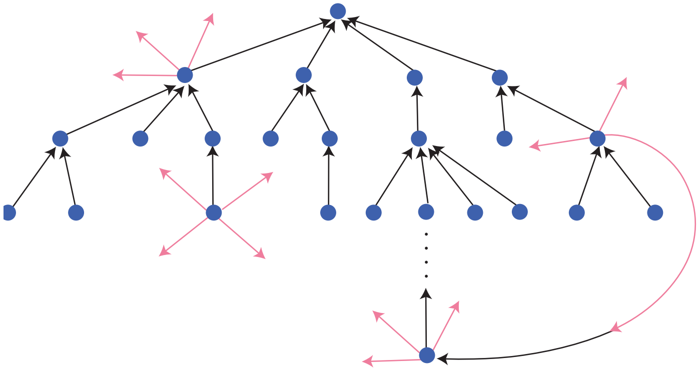
where $G_{r,s}$, $r, s \in \{1, \ldots, K\}$ is a distribution on $[0, H]$ for some constant H, independently of everything else.
- We construct the weights according to:

$$C(i,j) = \frac{cB_{ij}1(j \to i)}{\sum_{r=1}^{n} B_{ir}1(r \to i)},$$

if $D_i^- = \sum_{j=1}^{n} 1(j \to i) > 0$, and $C(i,j) \equiv 0$ otherwise.

Local tree-like behavior

- Consider a directed random graph $G_n = (V_n, E_n; \mathscr{A}_n)$ from any of the models mentioned earlier.
- Choose I_n uniformly in V_n and **explore** its in-component.



Local weak limit of locally tree-like random graphs

- **Local weak limits** characterize the local neighborhood of vertices.
- Unsurprisingly, for locally tree-like random (directed) graphs, the local weak limit is a (marked) **branching process**.
- For the dSBM with K communities the local weak limit is a K-type marked Galton-Watson tree.
- The directed configuration model and the rank-1 degree corrected dSBM have a single-type marked Galton-Watson tree as their local weak limit.

Back to the opinion model

- Recall $\nu(\mathbf{x})$ denotes the conditional distribution of $W_i^{(0)}$ given that its vertex **mark** is $\mathbf{X}_i = \mathbf{x}$.
- Suppose $d > 0$ and

$$d_1(\nu(\mathbf{x}), \nu(\tilde{\mathbf{x}})) \leq K\rho(\mathbf{x}, \tilde{\mathbf{x}}),$$

  for some $K < \infty$, and d_1 the Wasserstein metric of order 1.
- **Note:** $d > 0$ ensures that the map defining the recursion

$$R_i^{(t+1)} = \sum_{j=1}^{n} C(i,j) R_j^{(t)} + W_i^{(t)} + (1 - c - d) R_i^{(t)}, \qquad i \in V,$$

  is strictly contracting, making $R_i^{(t)}$ a **local** function of vertex i.

Sparse approximation

- For each $i \in V_n$ and each $t \geq 1$, let $\mathcal{T}_i^{(t)}(\boldsymbol{\mathcal{X}})$ denote the coupled depth-t marked branching tree rooted at vertex i and having the distribution of the local weak limit of $G_n = (V_n, E_n; \mathscr{A}_n)$.
- **Note:** It is possible to couple all n graph explorations with their local weak limits simultaneously.
- For each $i \in V_n$ and each $k \geq 1$ let $\mathcal{R}_{\emptyset(i)}^{(t)}$ denote the opinion at time t of the root $\emptyset(i)$ of $\mathcal{T}_i^{(t)}(\boldsymbol{\mathcal{X}})$, computed according to our model.
- The vector $\boldsymbol{\mathcal{R}}^{(t)} = (\mathcal{R}_{\emptyset(1)}^{(t)}, \ldots, \mathcal{R}_{\emptyset(n)}^{(t)})'$ does **NOT** have **independent components**.
- **Note:** For semi-sparse and dense graphs the corresponding opinion vector has independent components.

Sparse approximation... cont.

- **Theorem:** (Lin-OC '23-'25) For θ_n a constant and any fixed $t \geq 1$,

$$\lim_{n\to\infty} \max_{0\leq r\leq t} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_n\left[\left|R_i^{(t)} - \mathcal{R}_{\emptyset(i)}^{(t)}\right|\right] = 0,$$

and for any bounded and continuous function $f : \mathbb{R}^{t+1} \to \mathbb{R}$,

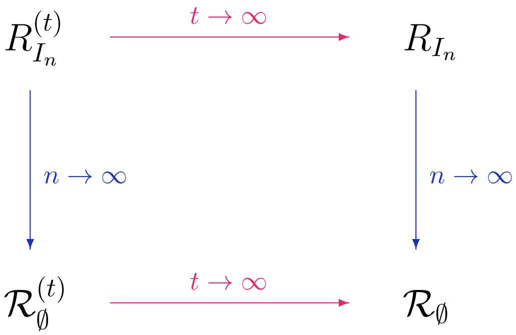
$$\frac{1}{n}\sum_{i=1}^{n} f(R_i^{(0)}, \dots, R_i^{(t)}) \xrightarrow{P} E\left[f(\mathcal{R}_\emptyset^{(0)}, \dots, \mathcal{R}_\emptyset^{(t)})\right], \qquad n \to \infty.$$

Moreover, if $\mathbf{R} = (R_1, \dots, R_n)'$ is distributed according to the stationary distribution of $\{\mathbf{R}^{(t)} : t \geq 0\}$, then, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{n}\sum_{i=1}^{n} f(R_i) \xrightarrow{P} E\left[f(\mathcal{R}_\emptyset)\right], \qquad n \to \infty.$$

Commuting diagram

- Let I_n denote a uniformly chosen vertex in $G_n = (V_n, E_n; \mathscr{A}_n)$.
- The theorem shows the following commuting diagram.



Remarks

- When the local weak limit is a K-type marked Galton-Watson process, the random variables

$$\mathcal{Y}^{(j)} \stackrel{\mathcal{D}}{=} (\mathcal{R}_\emptyset | \mathcal{J}_\emptyset = j),$$

where $\mathcal{J}_\emptyset \in \{1, \ldots, K\}$ is the community label of the root $\emptyset$, satisfy a **system of distributional fixed-point equations**.

- These equations allow us to compute

$$E[\mathcal{Y}^{(j)}] \qquad \text{and} \qquad \mathrm{Var}(\mathcal{Y}^{(j)})$$

for each $j \in \{1, \ldots, K\}$.

- Explicit formulas for conditional means and conditional variances in terms of only the limiting vertex marks are available.
- **Observation:** these are enough to characterize consensus and polarization, as well as to study the effects of cognitive biases.

The DeGroot model

- The DeGroot model does not define a strict contraction.
- The process $\{\mathbf{R}^{(t)} : t \geq 0\}$ has a limiting distribution that depends on the initial opinion vector $\mathbf{R}^{(0)}$.
- Consider the synchronous model.
- **Idea:** Perturb the model to obtain a strict contraction, i.e., approximate

$$\mathbf{R}^{(t+1)} = C\mathbf{R}^{(t)}$$

with

$$\tilde{\mathbf{R}}^{(t+1)} = (1-d)C\tilde{\mathbf{R}}^{(t)} + d\mathbf{1}$$

for some $d \in (0, 1)$.

- Choose $d = d(n)$ and $t = t(n)$ so that

$$(1-d)^t \to 1, \qquad n \to \infty$$

Locality for the DeGroot model

- Let $\tilde{\mathcal{R}}_{\emptyset}^{(t)}$ denote the value of the opinion of the root node in $\mathcal{T}^{(t)}(\boldsymbol{\mathcal{X}})$, the local weak limit of $\{G_n : n \geq 1\}$, of the perturbed process.
- Let $\mathcal{R}_{\emptyset}^{(t)}$ denote the value of the opinion of the root node in $\mathcal{T}^{(t)}(\boldsymbol{\mathcal{X}})$ of the original (unperturbed) process.
- Argue that
$$R_{I_n}^{(t)} \approx \tilde{R}_{I_n}^{(t)} \approx \tilde{\mathcal{R}}_{\emptyset}^{(t)} \approx \mathcal{R}_{\emptyset}^{(t)}$$
- Finally, analyze the distribution of $\lim_{t\to\infty} \mathcal{R}_{\emptyset}^{(t)}$.
- **Theorem:** (OC-Yu '25) The synchronous DeGroot model attains consensus as $t \to \infty$. Moreover, the value of the consensus is fully determined by $\mathcal{T}(\boldsymbol{\mathcal{X}})$ and the distribution of $\mathbf{R}^{(0)}$.

Thank you for your attention.