Multiple Extremal Integrals

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Introduction

- Recall multiple Wiener-Itô integral (Wiener N 1938, Itô K 1951):

$$\int_{E^k} f\left(t_1, t_2, \ldots, t_k\right) dW\left(t_1\right) dW\left(t_2\right) \ldots dW\left(t_k\right). \tag{1}$$

  where, f vanishes on diagonals, and W is a Brownian random measure.

  Here, changing W to other additive random measures, such as Poisson, α-stable, fractional Brownian random measures and so on, gives rise to different types of multiple stochastic integrals.

- J Chen, S Bai (2025) extend the single extremal integrals, which was introduced by L De Haan (1987), S Stoev, MS Taqqu (2005) to multiple extremal integrals. A k-fold extremal integral is in the form of

$$I_k^e(f) := {}^e\!\!\int_{E^k} f(s_1, s_2, \ldots, s_k) M_\alpha(ds_1) M_\alpha(ds_2) \ldots M_\alpha(ds_k) \tag{2}$$

  where M_α is a α-Fréchet random sup measure.

Multiple Extremal Integrals

Let $(E, \mathcal{E}, \mu)$ be a measure space. M_α is a α-Fréchet random sup measure satisfying the following conditions:

- (independently scattered) For any collection of disjoint sets $A_j \in \mathcal{E}, 1 \leq j \leq n$, $n \in Z_+$, the random variables $M_\alpha\left(A_j\right), 1 \leq j \leq n$, are independent.
- (α-Fréchet marginal) For any $A \in \mathcal{E}$,

$$\mathbb{P}\left\{M_\alpha(A) \leq x\right\} = \exp\left\{-\mu(A)x^{-\alpha}\right\} \text{ for } x \in (0, \infty).$$

- (σ-maxitive) For any collection of sets $A_j \in \mathcal{E}, j \in \mathbb{Z}_+$, we have that

$$M_\alpha\left(\bigcup_{j\geq 1} A_i\right) = \max_{j\geq 1} M_\alpha\left(A_j\right) \text{ almost surely.}$$

Examples:

- Single extremal integral: Let $f = 2\mathbf{1}_A + 5\mathbf{1}_B$ with $A, B \in \mathcal{E}$ disjoint, then

$$I_1^e(f) = {\int_E}^e f(u)M_\alpha(du) := \max\{2M_\alpha(A), 5M_\alpha(B)\}.$$

- Multiple extremal integral: Set $f = 2\mathbf{1}_{A_1\times A_2} + 5\mathbf{1}_{B_1\times B_2}$ with $A_1, A_2, B_1, B_2 \in \mathcal{E}$ disjoint, then the double integral of f is

$$I_2^e(f) = {\int_{E^2}}^e f(u,s)M_\alpha(du)M_\alpha(ds) := \max\{2M_\alpha(A_1)M_\alpha(A_2), 5M_\alpha(B_1)M_\alpha(B_2)\}.$$

When does the extremal integral $I_k^e(f)$ exist?

- $I_k^e(f)$ for a general $f : E^k \mapsto [0, \infty]$ vanishing on the diagonals can be defined by simple function approximation.

Theorem

(Sufficient condition for integrability) Suppose $(E, \mathcal{E}, \mu)$ is a probability measure space. A sufficient condition for $I_k^e(f) < \infty$ a.s is the following:
When $k = 2$, we need

$$\int_{E^2} f(s,t)^\alpha \left(1 + \ln_+ \frac{f(s,t)}{\left(\int_E f(s,u)^\alpha \mu(du)\right)^{1/\alpha} \left(\int_E f(u,t)^\alpha \mu(du)\right)^{1/\alpha}}\right) \mu(ds)\mu(dt) < \infty.$$

When $k \geq 3$, we need

$$\int_{E^k} f^\alpha(\boldsymbol{u}) \left(1 + (\ln_+ f(\boldsymbol{u}))^{k-1}\right) \mu^k(d\boldsymbol{u}) < \infty.$$

Remark

The idea of its proof is related to the theory of multiple α-stable integrals (Kwapień and Woyczyński 1987, Samorodnitsky and Szulga 1989).

Theorem

A necessary condition for $I_k^e(f) < \infty$ is $\int_{E^k} f^\alpha(\boldsymbol{u})\mu^k(d\boldsymbol{u}) < \infty$.

Welcome to my poster for more properties of $I_k^e(f)$!