Hölder regularity for a class of nonlinear stochastic heat equations

Sudheesh Surendranath

University of Utah

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- Consider the stochastic PDE:

$$\frac{\partial u(t,x)}{\partial t} = (\mathcal{L}u)(t,x) + \sigma(u(t,x))\dot{F}(t,x) \quad \text{for} \quad (t,x) \in (0,\infty) \times \mathbb{R}^n$$

- $\mathcal{L} := L^2(\mathbb{R}^n)-$generator of Lévy process $\{X_t\}_{t>0} \implies \widehat{\mathcal{L}} = -\Psi(\xi)$.
- $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz, $|\sigma(x) - \sigma(y)| \lesssim |x - y|$.
- $\dot{F} : \Omega \to \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^n)$ is centered Gaussian with $\mathbb{E}\left(\dot{F}(\varphi_1), \dot{F}(\varphi_2)\right) = \int_0^\infty \int_{\mathbb{R}^n} (\varphi_1 * \widetilde{\varphi_2})(t,x)\,\Gamma(dx)\,dt$, for a non-negative definite measure Γ. Formally, $\mathbb{E}(\dot{F}(t,x)\dot{F}(s,y)) = \delta(t-s)\Gamma(x-y)$.
- Bochner-Minlos-Schwarz theorem $\implies \Gamma = \widehat{\mu}$ with $\int_{\mathbb{R}^n} \frac{\mu(d\xi)}{(1+\|\xi\|^2)^k} < \infty$.
- [Dalang '99, Khoshnevisan-Foondun '13] A random-field solution $\{u(t,x)\}_{t>0, x\in\mathbb{R}^n}$ exists iff $\int_{\mathbb{R}^n} \frac{\mu(d\xi)}{1+\mathrm{Re}\,\Psi(\xi)} < \infty$.
- $u(t,x) = (P_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^n} P_{t-s}(x-y)\sigma(u(s,y))F(ds\,dy)$.

- [Walsh '86] In $d = 1$: for $\mathcal{L} = \Delta$ and $\Gamma = \delta_0$, $u \in \mathcal{C}_{\text{loc}}^{\frac{1}{4}-,\frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R})$.
- [Sanz-Solé '00,'02] For $\mathcal{L} = \Delta$: $\int_{\mathbb{R}^n} \frac{\mu(d\xi)}{(1+\text{Re}\,\Psi(\xi))^{1-\eta}} < \infty$ for some $\eta \in (0,1) \implies u \in \mathcal{C}_{\text{loc}}^{\frac{\eta}{4}-,\frac{\eta}{2}-}(\mathbb{R}_+ \times \mathbb{R}^n)$.
- [Khoshnevisan-Sanz-Solé '23] For $\sigma \equiv 1$: $\int_{\mathbb{R}^n} \frac{\|\xi\|^{2\eta}}{1+\text{Re}\,\Psi(\xi)} < \infty$ for some $\eta \in (0,1) \iff u \in \mathcal{C}_{\text{loc}}^{\frac{\eta}{4}-,\frac{\eta}{2}-}(\mathbb{R}_+ \times \mathbb{R}^n)$.
- Assume:
  - $\liminf_{\|\xi\|\to\infty} \frac{\text{Re}\,\Psi(\xi)}{\|\xi\|} = \text{l}_\infty > 0$ for $\text{l}_\infty \in (0,2)$
  - $\int_{\{\|\xi\|\le 1\}} \frac{\text{Re}\,\Psi(\xi)}{\|\xi\|^{n+\text{l}_0}}\, d\xi < \infty$ for $\text{l}_0 \in (0,1)$
  - $|\,\text{Im}\,\Psi(\xi)| \le \text{Re}\,\Psi(\xi)$ near 0.
- [S. '25]
  $\int_{\mathbb{R}^n} \frac{\mu(d\xi)}{(1+\text{Re}\,\Psi(\xi))^{1-\eta}} < \infty$ for some $\eta \in (0,1) \implies u \in \mathcal{C}_{\text{loc}}^{\frac{\alpha}{4}-,\frac{\alpha}{2}-}(\mathbb{R}_+ \times \mathbb{R}^n)$.
- Moreover, the conditions are equivalent.

Sketch of proof

- Moment estimates: $\mathbb{E}(\|X_t\|^{l_0}) \lesssim t^{\kappa} \quad \forall \kappa \in (0, \frac{l_0}{2})$.
- L^1-estimates:
- $$\int_{\mathbb{R}^n} |p_t(x+h) - p_t(x)|\, dx \lesssim \frac{\|h\|^{\alpha}}{t^{\eta}} \text{ for some } \alpha \in (0,1)$$
- $$\int_{\mathbb{R}^n} |p_{t+\varepsilon}(x) - p_t(x)|\, dx \lesssim \frac{\varepsilon^{\frac{\alpha}{2}}}{t^{\eta}} \text{ for some } \alpha \in (0,1)$$
- $$\sup_{[0,T]} \sup_{x \in \mathbb{R}^n} \mathbf{E}\left(|u(t,x+h) - u(t,x)|^{2p}\right) \lesssim \|h\|^{\alpha p} \qquad \forall p > 1$$
- $$\sup_{[0,T]} \sup_{x \in \mathbb{R}^n} \mathbf{E}\left(|u(t+\varepsilon,x) - u(t,x)|^{2p}\right) \lesssim |\varepsilon|^{\frac{\alpha p}{2}} \qquad \forall p > 1$$