Malliavin calculus on the stochastic heat equation and results on the density

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The stochastic problem

We apply Malliavin calculus on the one-dimensional stochastic heat equation on $[0, T] \times [0, 1]$ with Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x, u(t, x)) + \sigma(t, x, u(t, x))W(dt, dx).$$

- Pardoux-Zhang, Mueller-Nualart: If f, σ are Lipschitz with linear growth, $u(t, x)$ has a smooth density for all $t > 0, x \in (0, 1)$.
- The proof relies on a comparison principle and on the existence of negative moments for the solution to the linear stochastic heat equation.
- We prove the existence of a density for unbounded and Lipschitz f, σ, working directly with the nonlinear equation.
- We prove regularity results for the Malliavin derivative of u in the local sense and rely on a localization argument.

Strategy of the proof

- Construct a piecewise approximation of the solution u by u_n

$$u_n(t,x) = G_t u_0 + \int_0^t \int_0^1 G_{t-s}(x,y) H_n(|u_n(s,y)|) f(s,y,u_n(s,y))) dyds + \int_0^t \int_0^1 G_{t-s}(x,y) H_n(|u_n(s,y)|) \sigma(s,y,u_n(s,y)) W(dy,ds).$$

H_n cutoff functions, $H_n f, H_n \sigma$ bounded, $u_n = u$ on $\Omega_n = \{|u| \leq n\}$, $\Omega_n \uparrow \Omega$.

- Show that $u_n \in D_{1,2}$ and $u_n \in L_{1,2}$, so $u \in L_{1,2}^{loc}$.

$$\begin{aligned} D_{s,y} u_n(t,x) &= G_{t-s}(x,y) H_n(|u_n(s,y)|) \sigma(s,y,u_n(s,y)) \\ &+ \Big(\int_s^t \int_0^1 G_{t-\theta}(x,r) m(n)(\theta,r) D_{s,y} u_n(\theta,r) dr d\theta \\ &+ \int_s^t \int_0^1 G_{t-\theta}(x,r) \hat{m}(n)(\theta,r) D_{s,y} u_n(\theta,r) W(dr,d\theta) \Big) \\ &:= a(x,y,s,t) + b(x,y,s,t) \end{aligned}$$

Strategy of the proof

- Prove the estimates

$$\sup_{t\in[\hat{s}-\varepsilon,\hat{s}]}\sup_{x\in[0,1]}\mathbb{E}\left(\int_{\hat{s}-\varepsilon}^{\hat{s}}\int_0^1 |D_{s,y}u_n(t,x)|^2 dyds\right) < C\varepsilon^{\frac{1}{2}}$$

$$\mathbb{E}\left(\int_{t-\varepsilon}^{t}\int_0^1 b^2 dyds\right) \leq C\max\{\varepsilon^2, \varepsilon^{\frac{1}{2}}\int_{t-\varepsilon}^{t}\sum_{k=1}^{\infty} e^{-2k^2\pi^2(t-\theta)}\sin^2(k\pi x)d\theta\}.$$

- Use $|D_{s,y}u_n(t,x)|^2 \geq \frac{1}{2}a^2(x,y,s,t) - b^2(x,y,s,t)$ and the bounds on $\int_{t-\varepsilon}^{t}\int_0^1 a^2 dyds, \mathbb{E}\left(\int_{t-\varepsilon}^{t}\int_0^1 b^2 dyds\right)$ to show

$$P\Big(\omega\in\Omega_n : \int_0^t\int_0^1 |D_{s,y}u_n(t,x)|^2 dyds > 0\Big) = P(\Omega_n).$$

- Deduce

$$\int_0^t\int_0^1 |D_{s,y}u(t,x)|^2 dyds > 0 \text{ almost surely },$$

so the law of $u(t,x)$ has a density for all $t\in(0,T), x\in(0,1)$.