SPDEs on Metric Measure Spaces

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SPDEs on $\mathbb{R}^n$

$\partial_t u(t,x) = \Delta u(t,x) + \dot{\xi}(t,x), \quad u(0,\cdot) \equiv 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

$u(t,x) = \int_0^t \int_{\mathbb{R}^n} p_{t-s}(x,y)\xi(dy,ds)$

$\mathbb{E}\left[u(t,x)^2\right] < \infty \Leftrightarrow n < 2.$ ⟶ u is function valued only for $n < 2$.

Problem: $\partial_t u = \Delta u + \sigma(u)\dot{\xi}$ does not make sense for $n \geq 2$.

Question: What happens when $n \in (1,2)$?

$(\mathbb{R}^n, |\cdot|, dx)$: a space with a metric ($|\cdot|$) and a measure (dx).

$(\mathbb{R}^n, |\cdot|, dx) \longrightarrow (\mathbb{X}, d, m),$

$\Delta \longrightarrow \mathscr{L}$ (generator of Markov process Y on $\mathbb{X}$)

$\partial_t p_t(x,y) = \mathscr{L}_y p_t(x,y)$, usually with $p_t(x,y) > 0$ if $t > 0$.

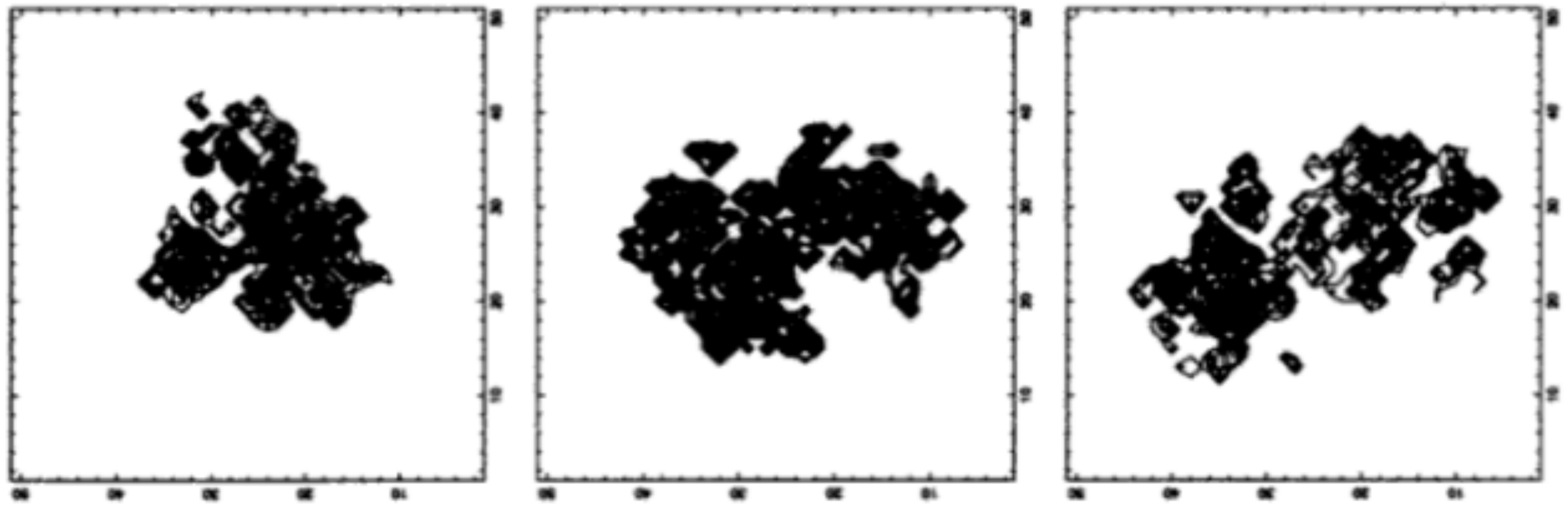
$u(t,x) = \int_0^t \int_{\mathbb{X}} p_{t-s}(x,y)\xi(dy,ds)$

A dimension constant $n \in [1,2)$

A surprising property

$$\partial_t u = \mathscr{L} u + \sqrt{u}\dot{\xi} \qquad (1)$$

Theorem: Compact Support Property (Fan, Sun, Y 2025+)
Under mild regularity conditions, there is a random field solution to (1) for any $u(0,\cdot) \in \mathscr{C}_c(\mathbb{X})$. If in addition that the Markov process generated by $\mathscr{L}$ is a diffusion and diffuses slow enough, then $\mathbb{P}\left(u(t,\cdot) \in \mathscr{C}_c^+(\mathbb{X}),\ \text{for all } t \geq 0\right) = 1$, given $u(0,\cdot) \in \mathscr{C}_c^+(\mathbb{X})$.



Applications:

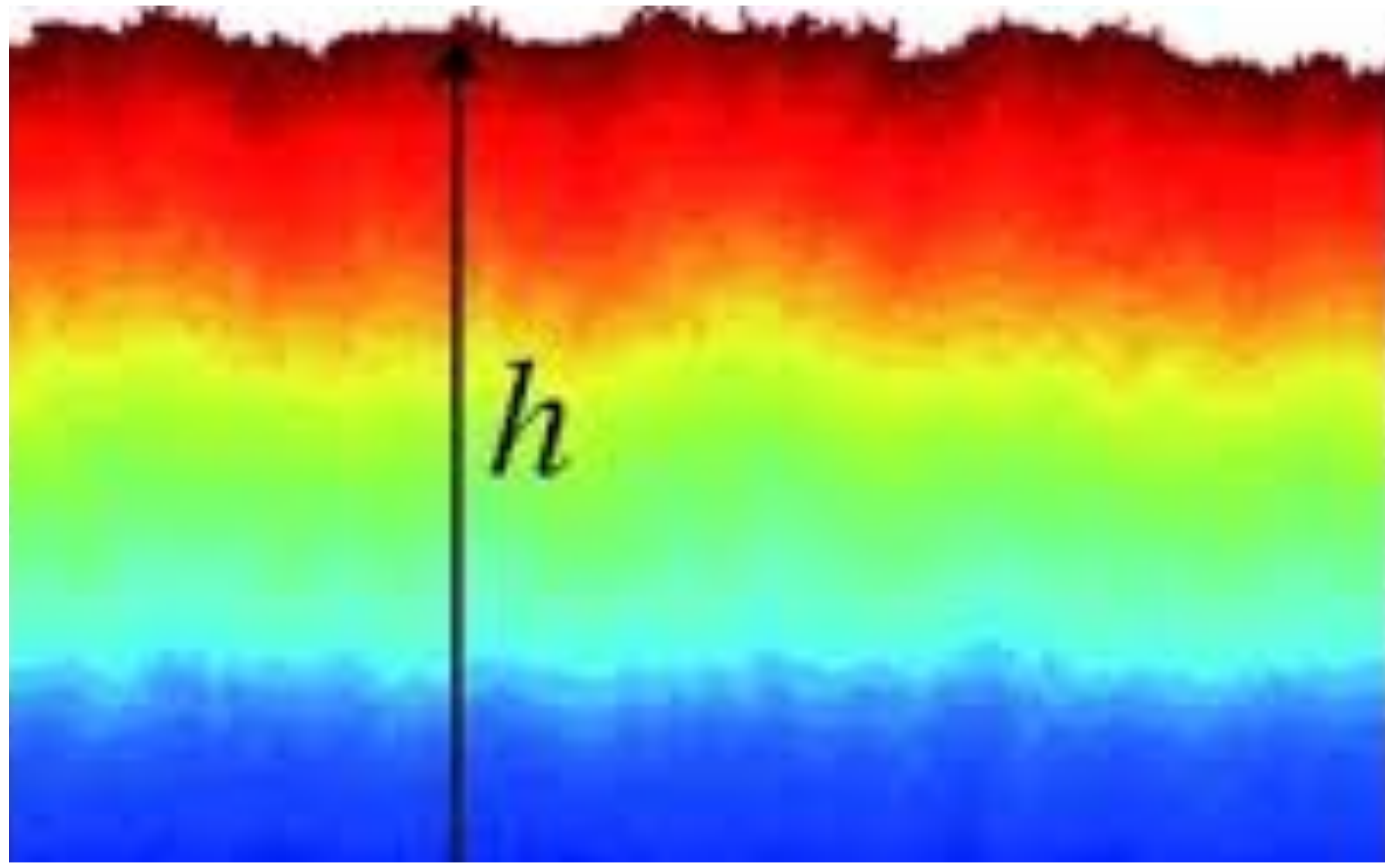
- random propagation speed of certain SPDEs on exotic spaces (Sierpinski gasket or continuum networks).
- Effects of the geometries on propagation speed.

A contrasting property

$$\partial_t u = \mathscr{L} u + u\dot{\xi} \qquad (2)$$

Theorem: Strict positivity (Fan, Sun, Y 2025+)

Under mild regularity conditions, there is a unique random field solution to (2) for any $u(0,\cdot) \in \mathscr{C}_c(\mathbb{X})$. If in addition the Markov process generated by $\mathscr{L}$ moves fast enough and $u(0,\cdot) \in \mathscr{C}_c^+(\mathbb{X}) \backslash \{\mathbf{0}\}$, then $\mathbb{P}\left(u(t,x) > 0, \text{ for all } t > 0, x \in \mathbb{X}\right) = 1$.



Application:

- KPZ equation on fractals via Cole-Hopf transformation.

Q: can one solve a equation with $n \in [2, 2+\epsilon)$?