

Improving the Lower Bound for the Union-closed Sets Conjecture via Conditionally IID Coupling

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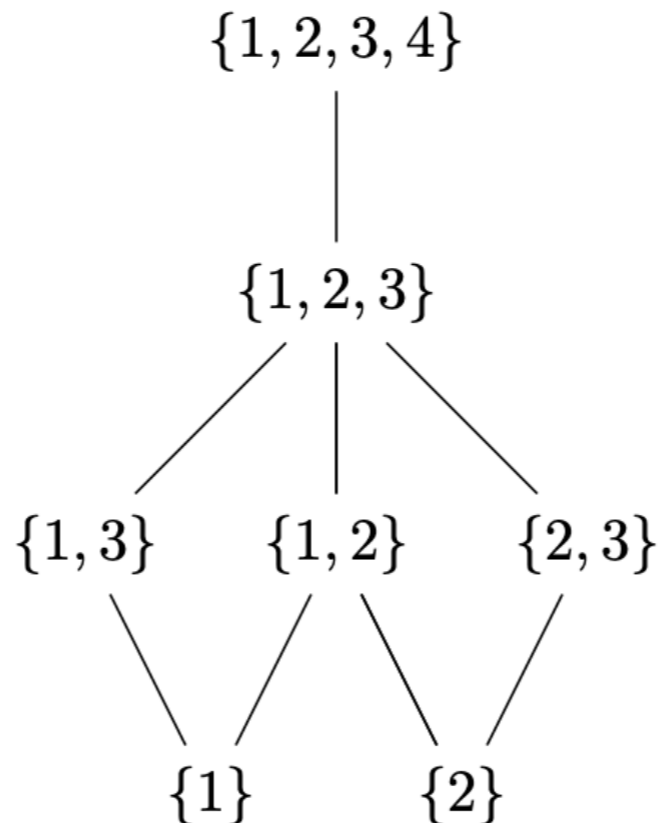
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The Union-closed Sets Conjecture (Frankl's Conjecture)

For any nonempty union-closed family of subsets of $[n] := \{1, 2, \dots, n\}$, there exists $i \in [n]$ that belongs to at least **half** of these subsets.

(union-closed means the union of any two sets from the family also belongs to the family)



Prior and New Results

- Classic combinatorial arguments of Knill and Wojick established lower bounds of order $\frac{1}{\log n}$ (in lieu of the conjectured lower bound of $1/2$).
- Justin Gilmer proved the first constant lower bound (0.01) by a probabilistic argument in Winter 2022. The idea is to establish an information-theoretic inequality involving an optimal coupling.
- A few days later, it was improved to $\frac{3-\sqrt{5}}{2} \approx 0.38197$, by Alweiss-Huang-Sellke, Chase-Lovett, and Sawin, using the same iid coupling of Gilmer but a sharper analysis.
- Sawin proposed a convex combination of the max entropy coupling and the iid coupling. The best constant obtainable this way is ≈ 0.38234 (Yu, Cambie).
- **This work:** We adopt a convex combination of a conditional iid coupling and the iid coupling, further improving the constant to at least 0.38271.

Information-theoretic Approach

- Let $X^n := \{X_1, \dots, X_n\}$ and Y^n be two equiprobable (on some subset of $\{0, 1\}^n$) and identically distributed binary vectors.
- The best constant for the union-closed sets conjecture is at least $c \in (0, 1/2)$ if $\mathbb{P}[X_i = 1] \leq c$ implies that

$$\max_{\text{coupling}} H(X^n \vee Y^n) > H(X^n) \quad (1)$$

since otherwise it violates the fact that the equiprobable distribution maximizes the entropy for a given support.

- The coupling of (X_i, Y_i) is constructed inductively to allow proof of (1) by induction. Greedily increasing the entropy of (X_i, Y_i) leads to greater uncertainty of the entropy of (X_{i+1}, Y_{i+1}) , so there is a tradeoff. Our improvement of c relies on a new coupling strategy.
- *“Improving the lower bound for the union-closed sets conjecture via conditionally IID coupling,”* J Liu, arXiv:2306.08824