Limit theorem for compensated Riemann sum and numerical SDEs

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if B is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, or if B is a general Gaussian process?

- Let y be the solution of the additive SDE:

$$dy_t = b(y_t)dt + \sigma(t)dB_t, \qquad y_0 = a \in \mathbb{R}^d,$$

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Question: What is the rate of convergence of the Euler scheme when $H < 1/2$? What if B is replaced by a general Gaussian process?

- These problems can be reduced to the study of a mix of weighted sum in the following form:

$$\sum_{k=0}^{n-1}(y_{t_k}h_k^{n,1} + y'_{t_k}h_k^{n,2} + \cdots + y_{t_k}^{(\ell-1)}h_k^{n,\ell})$$

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where $h_k^{n,i}$ is a functional of some underlying Gaussian process X and y is a process “controlled” by X (e.g. $y_t = f(X_t)$).

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$$\sum_{k=0}^{n-1} y_{t_k} h_k^{n,1} \sim O(1/n)^{2H} \qquad \text{and} \qquad \sum_{k=0}^{n-1} y'_{t_k} h_k^{n,2} \sim O(1/n)^{2H}.$$

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- Theorem (L. '24): We have $\mathcal{J}(y,h) \sim O(1/n)^{H+1/2}$, and $n^{H+1/2}\mathcal{J}(y,h) \to \int_0^T y_t dW_t$ as $n \to \infty$, where W is a Brownian motion independent of X.