Two Cute Results in Directed FPP

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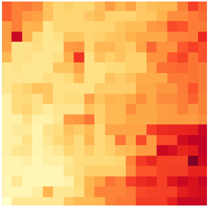
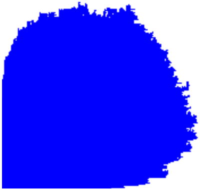
Definitions

First passage percolation (FPP) is a natural way to define a random (psuedo-)metric on $\mathbb{Z}^d$:

- Independent weights on edges $\omega(e)$,
- Weight of path is sum of weights on edges $L(\pi) = \sum_{e \in \pi} \omega(e)$,
- Distance between vertices is the minimum over paths $L(x, y) = \min_{\pi : x \sim y} L(\pi)$.





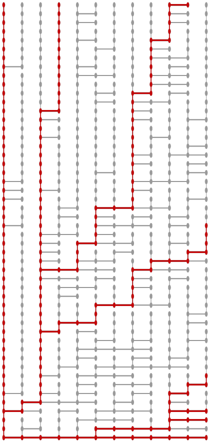
Many basic questions still open, such as: *what do the balls in this metric look like?* (We won't answer this.)

In general, "more restricted paths"≈"more tractable". We look at *directed* FPP.

Clustering of highways

Consider the tree of geodesics from the origin. At least some go off to ∞ – these are *highways*. An old problem is to describe the density of the highways. Settled for expected density but almost sure is harder.

Put i.i.d Exp(1) weights on horizontal edges and 0 on verticals. This model is *solvable* – explicit expressions for many quantities of interest.



Theorem

Let $A_k(n)$ be the proportion of highways in $[0, k] \times [0, n]$. Then

$$\liminf_{n\to\infty} \frac{A_k(n)}{n} = 0, \ \limsup_{n\to\infty} \frac{A_k(n)}{n} = 1 \text{ a.s.}$$

SJR limit shape decomposition

Balls in FPP converge a deterministic *limit shape*, the level sets of the *time constant*:

$$\lim_{n\to\infty} L(0,(nx,ny)) = f(x,y)\ \mathrm{a.s.}$$

The limiting ball is $B = \{f(x,y) \leq 1\}$.

In the SJR model each vertex has at most one non-zero weight on the edges coming *into* it.

Theorem

Let f_1 be the time constant for the model with the same horizontal weights but zero vertical weights, and B_1 the corresponding ball (and f_2, B_2 the same but with vertical weights). Then

$$B = B_1 \cap B_2.$$