Title

Global Geometry and the Multiplicative Stochastic Heat Equation

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Introduction

- We wish to study the equation

$$\left(\partial_t - \frac{1}{2}\triangle\right) u(t,x) = u(t,x)\cdot W, \quad t>0, x\in M, \quad (1)$$
$$u(0,x) = \mu$$

A solution is a random field $\{\, u(t,x)\,\}_{t>0,x\in M}$ where M is a compact Riemannian manifold and μ is a finite measure. We seek to answer the following two questions.

1. What regularity requirements on W is reasonable for (1) to be well-posed?
2. Do the moments of u grow exponentially in time?

Colored Noise and Well-Posedness

- We construct an intrinsic family of Gaussian noises $\{ W_\alpha \}_{\alpha \geq 0}$ where α is a regularity parameter.
- $\alpha = 0$ would be space-time white noise, which is well-known to not admit random field solutions in dimension $d \geq 2$.

Theorem (C-Ouyang '25+)

(1) *is has a unique solution for* $\alpha > \frac{d-2}{2}$ *if* M *has non-positive sectional curvature.*

- $\alpha > \frac{d-2}{2}$ is optimal and expected from the $\mathbb{R}^d$ literature.
- The non-positive curvature condition is used for its global geometry implications. We do not know if $\alpha > \frac{d-2}{2}$ will be the optimal if we remove it.

Behavior of Moments

Definition (Carmona-Molchanov'94)

For $p \geq 2$, define $L_p(x) := \lim_{t\uparrow+\infty} \frac{1}{t} \ln \mathbb{E}[|u(t,x)|^p]$. A solution $u(t,x)$ of (1) is **intermittent** if

- For all $p \geq 2$, $\sup_{x\in M} L_p(x) < +\infty$.
- $\inf_{x\in M} L_2(x) > 0$.

Theorem (C-Ouyang '25+)

If μ is a positive finite measure, u solving (1) on a compact manifold is intermittent by the above definition.

- The upper bound follows naturally from the proof of existence and uniqueness.
- The lower bound is new and uses the ergodicity of Brownian motion on compact manifolds.